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Measuring influence in command games

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Abstract. In the paper, we study a relation between command games proposed by Hu and Shapley and an influence model. We show that our framework of influence is more general than the framework of the command games. We define several influence functions which capture the command structure. These functions are compatible with the command games, in the sense that each commandable player for a coalition in the command game is a follower of the coalition under the command influence function. For some influence functions we define the command games such that the influence functions are compatible with these games. We show that not for all influence functions such command games exist. Moreover, we propose a more general definition of the influence index and show that some power indices, which can be used in the command games, coincide with some expressions of the weighted influence indices. We show exact relations between an influence function and a follower function, between a command game and commandable players, and between influence functions and command games. An example of the Confucian model of society is broadly examined.

JEL Classification: C7, D7

Keywords: influence function, follower, influence index, command game, commandable player, Shapley-Shubik index, Banzhaf index, Coleman indices, König-Bräuninger index

1 Introduction

Studying an interaction and an influence among players in voting situations exposes many challenging problems to be solved. In traditional framework of power indices (see e.g., [1–5, 11, 14–17, 19], see also [6] for an overview), neither interaction nor influence among the voters is assumed. The concept of the interaction or cooperation among players in a cooperative game is studied, for instance, in [7], where players in a coalition are said to exhibit a positive (negative) interaction when the worth of the coalition is greater (smaller) than the sum of the individual worths. The authors present an axiomatization of the interaction indices which are extensions of the Shapley and Banzhaf values.

Another approach is presented in [12, 13], where the command structure of Shapley [18] is applied to model players' interaction relations by simple games. For each player, boss sets and approval sets are introduced. While the boss sets are defined as the 'sets of individuals that the player must obey, regardless of his own judgment or desires', the 'consent of the approval set is sufficient to allow the player to act, if he wishes' [13]. Based on the boss and approval sets, a simple game called the command game for a player is built. Furthermore, an equilibrium authority distribution is formulated to which the Shapley-Shubik index is applied.

Coming still from a different direction is an approach proposed by Hoede and Bakker [10], where the authors talk explicitly about an influence between players in a social network. Players who are to make a yes-no decision, have their inclinations to say either 'yes' or 'no', but due to the influence by the others, they can decide differently from their inclinations. The model of Hoede and Bakker is the point of departure for our research on influence. In [8] we introduce and study weighted influence indices of a coalition on a player in a social network, and consider different influence functions. In [9] we extend the yes-no model to a multi-choice game, and assume that players have a totally ordered set of possible actions instead of just two actions 'yes' or 'no'. Each player has an inclination to choose one of the actions. Consequently, the generalized influence indices are investigated.

The aim of the present paper is to study a relation between the command games considered by Hu and Shapley [12, 13] and the influence functions and influence indices defined in [8, 9]. We show that our framework of influence is in a sense more general than the framework of command games. In the influence model, many different influence functions may be proposed. In the present paper, we define several influence functions which capture the command structure proposed by Hu and Shapley. These influence functions are compatible with the command games, in the sense that each commandable player for a coalition in the command game is a follower of the coalition under the command influence function. For some influence functions introduced in [8], we define the command games such that the influence functions are compatible with these games. In particular, we show that not for all influence functions such command games exist.

Moreover, we propose a more general definition of the influence index and show that some power indices, which can be used in the command games, coincide with some expressions of the weighted influence indices under the considered command influence functions.

The main results of the paper concern a relation between influence functions and command games. We show exact relations between an influence function and a follower function, between a command game and commandable players, and between influence functions and command games.

The structure of the paper is the following. Definitions of power indices we refer to in the paper are recapitulated in Section 2. In Section 3 the framework of the command games is briefly presented. The latter model of the relationship we search for, that is, the influence model, is summarized in Section 4. The core of the paper is presented in Sections 5, 6, and 7, where the relation between the influence model and the command games is investigated. In Section 5, we define several influence functions which are compatible with the command games. We show some relations between several power indices and the weighted influence indices under the command influence functions. In Section 6, we define the command games for some influence functions introduced in [8]. In Section 7, we show the relation between the influence functions and the command games. In Section 8, the relations between the influence model and the command games are illustrated by an example mentioned in [13], i.e., by the Confucian model of society. In Section 9, we conclude.

2 Power indices

First, we introduce several notations for convenience. Cardinality of sets $S, T, \ldots$ will be denoted by the corresponding lower case $s, t, \ldots$. We omit braces for sets, e.g., $\{k, m\}$, $N \setminus \{j\}$, $S \cup \{j\}$ will be written km, $N \setminus j$, $S \cup j$, etc.

A $(0,1)$-*game* is a pair (N, v), where $N = \{1, ..., n\}$ is the set of players, and a function $v : 2^N \to \{0,1\}$ satisfying $v(\emptyset) = 0$ is the characteristic function. A nonempty subset of N is called a *coalition*. A *simple game* is a $(0,1)$-game such that v is not identically equal to 0 and is monotonic, i.e., $v(S) \leq v(T)$ whenever $S \subseteq T$. A coalition S is *winning* if $v(S) = 1$, and is *loosing* if $v(S) = 0$. A game is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ whenever $S \cap T = \emptyset$. Let SG_n denote the set of all simple superadditive n-person games. Player k is a *swing* in a winning coalition S if his removal from the coalition makes it loosing, i.e., if $v(S) = 1$ and $v(S \setminus k) = 0$. A *minimal winning coalition* is a winning coalition in which all players are swings.

A *power index* is a function $\phi : SG_n \to \mathbb{R}^n$ which assigns to each $(N, v) \in SG_n$ a vector $\phi(N, v) = (\phi_1(N, v), ..., \phi_n(N, v))$.

The *Shapley-Shubik index* [19] of player $k \in N$ in a game (N, v) is defined by

$$Sh_k(N, v) = \sum_{S \subseteq N : k \in S} \frac{(n-s)!(s-1)!}{n!} (v(S) - v(S \setminus k)). \tag{1}$$

The *non-normalized Banzhaf index (the absolute Banzhaf index, [1])* of player $k \in N$ in a game (N, v) is defined by

$$Bz_k(N, v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N : k \in S} (v(S) - v(S \setminus k)), \tag{2}$$

and the *normalized Banzhaf index* $\widetilde{Bz}$ is given by

$$\widetilde{Bz}_k(N, v) = \frac{Bz_k(N, v)}{\sum_{j \in N} Bz_j(N, v)}. \tag{3}$$

The *Coleman 'power of a collectivity to act'* [2], [3] in a game (N, v) is defined by

$$A(N, v) = \frac{\sum_{S \subseteq N} v(S)}{2^n}. \tag{4}$$

The *Coleman index 'to prevent action'* [2], [3] of player $k \in N$ in a game (N, v) is defined by

$$Col_k^P(N, v) = \frac{\sum_{S \subseteq N : k \in S} (v(S) - v(S \setminus k))}{\sum_{S \subseteq N} v(S)}. \tag{5}$$

The *Coleman index 'to initiate action'* [2], [3] of player $k \in N$ in a game (N, v) is defined by

$$Col_k^I(N, v) = \frac{\sum_{S \subseteq N : k \notin S} (v(S \cup k) - v(S))}{2^n - \sum_{S \subseteq N} v(S)}. \tag{6}$$

The *König-Bräuninger inclusiveness index* [15] of player $k \in N$ in a game (N, v) is defined by

$$KB_k(N, v) = \frac{\sum_{S \subseteq N : k \in S} v(S)}{\sum_{S \subseteq N} v(S)} \tag{7}$$

3 The command games

We recapitulate briefly the main concepts concerning the command games introduced by Hu and Shapley [12, 13]. Let $N = \{1, ..., n\}$ be the set of players (voters). For $k \in N$ and $S \subseteq N \setminus k$:

- S is a *boss set* for k if S determines the choice of k;
- S is an *approval set* for k if k can act with an approval of S.

It is assumed that any superset (in $N \setminus k$) of a boss set is a boss set.

For each $k \in N$, a simple game $(N, \mathcal{W}_k)$ is built, called the *command game for* k, where the set of winning coalitions is

$$\mathcal{W}_k := \{S \mid S \textit{ is a boss set for } k\} \cup \{S \cup k \mid S \textit{ is a boss or approval set for } k\}. \tag{8}$$

We can recover the boss sets $Boss_k$

$$Boss_k = \{S \subseteq N \setminus k \mid S \in \mathcal{W}_k\} = \mathcal{W}_k \cap 2^{N \setminus k} \tag{9}$$

and the approval sets App_k

$$App_k = \{S \subseteq N \setminus k \mid S \cup k \in \mathcal{W}_k \textit{ but } S \notin \mathcal{W}_k\}. \tag{10}$$

We have $Boss_k \cap App_k = \emptyset$. In particular, if $App_k = 2^{N \setminus k}$, then k is called a *free agent*: he needs no approval (since $\emptyset \in App_k$), and nobody can boss him (since $Boss_k = \emptyset$). If $App_k = \emptyset$, then k is called a *cog*.

Given the command games $\{(N, \mathcal{W}_k) \mid k \in N\}$, for any coalition $S \subseteq N$, the *set* $\omega(S)$ *of all members that are 'commandable' by* S is defined by:

$$\omega(S) := \{k \in N \mid S \in \mathcal{W}_k\}. \tag{11}$$

We have: $\omega(\emptyset) = \emptyset$, $\omega(N) = N$, and $\omega(S) \subseteq \omega(S')$ whenever $S \subset S'$.

An *authority distribution* $\pi = (\pi_1, ..., \pi_n)$ over an organization $(N, \{(N, \mathcal{W}_k) \mid k \in N\})$ satisfies

$$\pi_k \geq 0 \textit{ for any } k \in N \textit{ and } \sum_{k \in N} \pi_k = 1.$$

The *power transition matrix* of the organization is the stochastic matrix $P = [P(j,k)]_{j,k=1}^n$ such that

$$P(j,k) := Sh_k(N, \mathcal{W}_j), \tag{12}$$

where $Sh_k(N, \mathcal{W}_j)$ is the Shapley-Shubik index of player k in the command game for player j. The authority distribution π is assumed to satisfy the *authority equilibrium equation* given by

$$\pi = \pi P, \textit{ i.e.}, \; \pi_k = \sum_{j \in N} \pi_j P(j,k), \; \forall k \in N. \tag{13}$$

In the sense of a political counterbalance equilibrium, $\pi_j P(j,k)$ is the authority flowing from j to k. Hence, π_k is the sum of these authorities flowing to k.

Let $P^2 = PP$, $P^3 = PPP$, etc. Player k is said to *influence* j if $P^k(j,k) > 0$ for some $k > 0$, and j, k *communicate* if they influence each other ([13]). A coalition (or an organization) is said to be *irreducible* if any two members of the coalition (or the organization) communicate. Another interpretation of π_k is the long-run influence of player k on other players. If the organization is irreducible and aperiodic, then

$$\pi_k = \lim_{t\to\infty} P^t(j,k),$$

and the limit is independent of the choice of j. Hu and Shapley [13] call it the 'uniform ultimate influence' of k to other members of the organization.

4 The influence model

4.1 Direct versus opposite influence

In this section, the main concepts introduced in [8] are summarized. Let $N := \{1, ..., n\}$ be the set of players (agents, actors, voters). The players are to make a yes-no decision. Each player has an inclination either to say 'yes' (denoted by $+1$) or 'no' (denoted by -1). Let $i = (i_1, ..., i_n)$ denote an *inclination vector* and $I := \{-1, +1\}^n$ be the set of all inclination vectors. Players may influence each other, and as a consequence of the influence, the final decision of a player may be different from his original inclination. Each inclination vector $i \in I$ is therefore transformed into a *decision vector* $Bi = ((Bi)_1, ..., (Bi)_n)$, where $B : I \to I$ is the influence function. Let $\mathcal{B}$ denote the set of all influence functions.

We introduce for any $S \subseteq N$ the set

$$I_S := \{i \in I \mid \forall k, j \in S\ [i_k = i_j]\}, \tag{14}$$

and $I_k := I$, for any $k \in N$. We denote by i_S the value i_k for some $k \in S$, $i \in I_S$. For each $S \subseteq N$ and $j \in N$, we define the set $I_{S\to j}$ of all inclination vectors of potential direct influence of S on j, and the set $I^*_{S\to j}(B)$ of all inclination vectors of observed direct influence of S on j under given $B \in \mathcal{B}$:

$$I_{S\to j} := \{i \in I_S \mid i_j = -i_S\} \tag{15}$$

$$I^*_{S\to j}(B) := \{i \in I_{S\to j} \mid (Bi)_j = i_S\}. \tag{16}$$

Similarly, for each $S \subseteq N$ and $j \in N$ we introduce the set $I^{op}_{S\to j}$ of all inclination vectors of potential opposite influence, and the set $I^{*op}_{S\to j}(B)$ of all inclination vectors of observed opposite influence of a coalition on a player:

$$I^{op}_{S\to j} := \{i \in I_S \mid i_j = i_S\} \tag{17}$$

$$I^{*op}_{S\to j}(B) := \{i \in I^{op}_{S\to j} \mid (Bi)_j = -i_S\}. \tag{18}$$

For each $S \subseteq N$, $j \in N \setminus S$ and $i \in I_S$, we introduce a *weight* $\alpha_i^{S\to j} \in [0,1]$ *of influence* of coalition S on $j \in N$ under the inclination vector $i \in I_S$. We assume that for each $S \subseteq N$ and $j \in N \setminus S$, there exists $i \in I_{S\to j}$ such that $\alpha_i^{S\to j} > 0$, and there exists $i \in I^{op}_{S\to j}$ such that $\alpha_i^{S\to j} > 0$.

Given $B \in \mathcal{B}$, for each $S \subseteq N$, $j \in N \setminus S$, the *weighted direct influence index* of coalition S on player j is defined as

$$d_\alpha(B, S \to j) := \frac{\sum_{i\in I^*_{S\to j}(B)} \alpha_i^{S\to j}}{\sum_{i\in I_{S\to j}} \alpha_i^{S\to j}} \in [0,1] \tag{19}$$

and the *weighted opposite influence index* of coalition S on player j is defined as

$$d_\alpha^{op}(B, S \to j) := \frac{\sum_{i \in I_{S\to j}^{*op}(B)} \alpha_i^{S\to j}}{\sum_{i \in I_{S\to j}^{op}} \alpha_i^{S\to j}} \in [0,1]. \tag{20}$$

Let $\emptyset \neq S \subseteq N$ and $B \in \mathcal{B}$. We define the *set $F_B(S)$ of followers* of S under B as

$$F_B(S) := \{j \in N \mid \forall i \in I_S \; [(Bi)_j = i_S]\}. \tag{21}$$

We assume $F_B(\emptyset) = \emptyset$. We have, in particular,

$$d_\alpha(B, S \to j) = 1, \; \forall j \in F_B(S) \setminus S. \tag{22}$$

Assume F_B is not identically the empty set. The *kernel* of B, that is, the set of 'true' influential coalitions, is the following collection of sets:

$$\mathcal{K}(B) := \{S \in 2^N \mid F_B(S) \neq \emptyset, \text{ and } S' \subset S \Rightarrow F_B(S') = \emptyset\}. \tag{23}$$

4.2 A general case of influence

Next, we generalize the concept of influence to be able to comprise all imaginable cases of influence, in particular, the direct influence, the opposite influence, an influence of a coalition on its member, etc. The point of departure will be the set I_S defined in (14). In other words, we assume that a coalition may influence a player only in situations when all members of the coalition have the same inclination. Furthermore, an observed influence of a coalition S on a player j, which depends now on $\lambda \in \{i_j, -i_j, i_S, -i_S, +1, -1\}$, takes place if $(Bi)_j = \lambda$. Hence, the *set $I_{S\to j,\lambda}(B)$ of all inclination vectors of influence of S on j under B* is defined as

$$I_{S\to j,\lambda}(B) := \{i \in I_S \mid (Bi)_j = \lambda\}. \tag{24}$$

Definition 1 *Given $B \in \mathcal{B}$, for each $S \subseteq N$, $j \in N$, the weighted influence index of coalition S on player j is defined as*

$$\psi_{\alpha,\lambda}(B, S \to j) := \frac{\sum_{i \in I_{S\to j,\lambda}(B)} \alpha_i^{S\to j}}{\sum_{i \in I_S} \alpha_i^{S\to j}}. \tag{25}$$

The weighted direct influence index is recovered as follows:

$$\psi_{\alpha,\lambda}(B, S \to j) = d_\alpha(B, S \to j) \quad if \quad \lambda = -i_j \quad and \quad \alpha_i^{S\to j} = 0 \quad for \quad i_j = i_S. \tag{26}$$

The weighted opposite influence index is recovered as:

$$\psi_{\alpha,\lambda}(B, S \to j) = d_\alpha^{op}(B, S \to j) \quad if \quad \lambda = -i_j \quad and \quad \alpha_i^{S\to j} = 0 \quad for \quad i_j = -i_S. \tag{27}$$

The essence of both the direct influence and the opposite influence is that the decision of an influenced player is different from his original inclination ($\lambda = -i_j$). The only difference between these two types of influence lies in the inclination vectors we consider: while in the direct influence we observe only situations in which the inclinations of a player and an influencing coalition are different from each other ($\alpha_i^{S\to j} = 0$ for $i_j = i_S$), in case of the opposite influence we take into account only situations in which the inclinations of a player and an influencing coalition are the same ($\alpha_i^{S\to j} = 0$ for $i_j = -i_S$).

The set of followers and the kernel are defined like in (21) and (23), respectively.

4.3 Influence functions

We recapitulate some influence functions defined in [8] that we refer to in the present paper.

(i) **The Majority function** - Let $n \geq t > \lfloor \frac{n}{2} \rfloor$, and introduce for any $i \in I$ the set $i^+ := \{k \in N \mid i_k = +1\}$. The *majority influence function* $\mathsf{Maj}^{[t]} \in \mathcal{B}$ is defined by

$$\mathsf{Maj}^{[t]} i := \begin{cases} 1_N, & \text{if } |i^+| \geq t \\ -1_N, & \text{if } |i^+| < t \end{cases}, \quad \forall i \in I. \tag{28}$$

We have for each $S \subseteq N$:

$$F_{\mathsf{Maj}^{[t]}}(S) = \begin{cases} N, & \text{if } \ s \geq t \\ \emptyset, & \text{if } \ s < t \end{cases} \tag{29}$$

$$\mathcal{K}(\mathsf{Maj}^{[t]}) = \{S \subseteq N \mid |S| = t\}. \tag{30}$$

(ii) **The Guru function** - Let $\widetilde{k} \in N$ be a particular player called the guru. The *guru influence function* $\mathsf{Gur}^{[\widetilde{k}]} \in \mathcal{B}$ is defined by

$$(\mathsf{Gur}^{[\widetilde{k}]} i)_j = i_{\widetilde{k}}, \quad \forall i \in I, \quad \forall j \in N. \tag{31}$$

We have for each $S \subseteq N$:

$$F_{\mathsf{Gur}^{[\widetilde{k}]}}(S) = \begin{cases} N, & \text{if } \widetilde{k} \in S \\ \emptyset, & \text{if } \widetilde{k} \notin S \end{cases} \tag{32}$$

$$\mathcal{K}(\mathsf{Gur}^{[\widetilde{k}]}) = \{\widetilde{k}\}. \tag{33}$$

(iii) **The identity function** $\mathsf{Id} \in \mathcal{B}$ is defined by

$$\mathsf{Id} i = i, \quad \forall i \in I. \tag{34}$$

We have for each $S \subseteq N$, $F_{\mathsf{Id}}(S) = S$. The kernel is $\mathcal{K}(\mathsf{Id}) = \{\{k\}, k \in N\}$.

(iv) **The reversal function** $-\mathsf{Id} \in \mathcal{B}$ is simply the opposite of the identity function:

$$-\mathsf{Id} i = -i, \quad \forall i \in I. \tag{35}$$

We have for each $S \subseteq N$, $F_{-\mathsf{Id}}(S) = \emptyset$. The kernel is $\mathcal{K}(-\mathsf{Id}) = \emptyset$.

5 Command influence functions

We can apply our model of influence to the framework of the command games. First, we propose several influence functions, called here the *command influence functions* that are *compatible* with the command games, in the sense that each commandable player for a coalition in the command game is a follower of the coalition under the command influence function.

Definition 2 *Let $\{(N, \mathcal{W}_k) \mid k \in N\}$ be the command games, and $\omega(S)$ be the set of all players commandable by S as defined in (11). The influence function B is compatible with the command games $\{(N, \mathcal{W}_k) \mid k \in N\}$ if*

$$\omega(S) \subseteq F_B(S), \ \forall S \subseteq N, \tag{36}$$

where $F_B(S)$ is the set of followers of S under the influence function B.

For each set of the command games, the necessary condition for the existence of the influence function which is compatible with the command games is as follows:

Proposition 1 *Let $\{(N, \mathcal{W}_k) \mid k \in N\}$ be the command games. If there exists an influence function which is compatible with these command games, then the following holds:*

$$S \cap S' \neq \emptyset, \ \forall k \in N \ \forall S, S' \in \mathcal{W}_k. \tag{37}$$

Proof: Suppose that there exists the influence function B which is compatible with the command games $\{(N, \mathcal{W}_k) \mid k \in N\}$, and the condition (37) does not hold. Hence, from (36), $\omega(S) \subseteq F_B(S)$, for each $S \subseteq N$. Moreover, since (37) does not hold, there exist $k \in N$, and $S, S' \in \mathcal{W}_k$ such that $S \cap S' = \emptyset$. Since $S, S' \in \mathcal{W}_k$, we have $k \in \omega(S)$ and $k \in \omega(S')$, which by virtue of (36) implies $k \in F_B(S)$ and $k \in F_B(S')$. From (21), $(Bi)_k = i_S$ for each $i \in I_S$, and $(Bi)_k = i_{S'}$ for each $i \in I_{S'}$. Take $i \in I_S \cap I_{S'}$ such that $i_S \neq i_{S'}$. Hence, we have $(Bi)_k = i_S \neq i_{S'} = (Bi)_k$, contradiction. ■

In this paper, we apply the power indices (recapitulated in Section 2) to the command games, and we show that these power indices coincide with some expressions of the weighted influence indices under the command influence functions. We consider the Shapley-Shubik index matrix as defined in (12), i.e.,

$$Sh = [Sh_k(N, \mathcal{W}_j)]_{j,k=1}^{n} \tag{38}$$

where $Sh_k(N, \mathcal{W}_j)$ is the Shapley-Shubik index of player k in the command game for j.

We create also the Banzhaf index matrix

$$Bz = [Bz_k(N, \mathcal{W}_j)]_{j,k=1}^{n} \tag{39}$$

where $Bz_k(N, \mathcal{W}_j)$ is the Banzhaf index of player k in the command game for j, and matrices related to the Coleman indices and the König-Bräuninger index. Let

$$A = [A(N, \mathcal{W}_j)]_{j \in N} \tag{40}$$

$$Col^P = [Col_k^P(N, \mathcal{W}_j)]_{j,k=1}^{n} \tag{41}$$

$$Col^I = [Col_k^I(N, \mathcal{W}_j)]_{j,k=1}^{n} \tag{42}$$

$$KB = [KB_k(N, \mathcal{W}_j)]_{j,k=1}^{n} \tag{43}$$

where $A(N, \mathcal{W}_j)$ is the Coleman power of a collectivity to act in the command game for player j, $Col_k^P(N, \mathcal{W}_j)$ / $Col_k^I(N, \mathcal{W}_j)$ is the Coleman index to prevent action / the Coleman index to initiate action of player k in the command game for j, and $KB_k(N, \mathcal{W}_j)$ is the König-Bräuninger inclusiveness index of k in the command game for j.

As will be shown in this section, there exist relations between these indices in the command games, and the weighted influence indices under the command influence functions.

5.1 An influence function with abstention

The first command influence function we propose is related to an extended model of influence (see [9]), in which players have an ordered set of possible actions. In the present model, each player has an inclination either to say 'yes' (denoted by +1) or 'no' (denoted by −1), but he has three options to make his decision: 'yes', 'no', or 'to abstain' (denoted by 0). The command influence function is defined as follows.

Definition 3 *Given the command games $\{(N, \mathcal{W}_k) \mid k \in N\}$, the command influence function $\mathsf{Com} \in \mathcal{B}$ is defined for each $k \in N$ and $i \in I$ by*

$$(\mathsf{Com}i)_k := \begin{cases} +1, & \textit{if } \{j \in N \mid i_j = +1\} \in \mathcal{W}_k \\ -1, & \textit{if } \{j \in N \mid i_j = -1\} \in \mathcal{W}_k \\ 0, & \textit{otherwise} \end{cases}. \tag{44}$$

According to the command influence function Com, for each voter k and each inclination vector, if all players with the same inclination forms a winning coalition in his command game, the voter k follows the inclination of this winning coalition. Otherwise, that is, if none of the two coalitions with all members having the same inclination is winning, the voter k simply abstains.

Proposition 2 *Let $\{(N, \mathcal{W}_k) \mid k \in N\}$ be the command games, and $\omega(S)$ be the set of all players that are 'commandable' by S, as defined in (11). We have*

$$F_{\mathsf{Com}}(S) = \omega(S), \ \forall S \subseteq N, \tag{45}$$

where $F_{\mathsf{Com}}(S)$ is the set of followers of S under the command influence function Com.

Proof: $F_{\mathsf{Com}}(\emptyset) = \emptyset = \omega(\emptyset)$. Take an arbitrary $S \subseteq N$. Suppose that $F_{\mathsf{Com}}(S) \not\subseteq \omega(S)$. Hence, there exists $k \in F_{\mathsf{Com}}(S)$ such that $k \notin \omega(S)$, and therefore $(\mathsf{Com}i)_k = i_S$ for all $i \in I_S$, and $S \notin \mathcal{W}_k$. Take $i \in I_S$ such that $i_S = +1$, and $i_j = -1$ for each $j \notin S$. Hence, $(\mathsf{Com}i)_k = i_S = +1$, but since $S \notin \mathcal{W}_k$ we have also either $(\mathsf{Com}i)_k = -1$ or $(\mathsf{Com}i)_k = 0$, contradiction.
Suppose now that $\omega(S) \not\subseteq F_{\mathsf{Com}}(S)$. Hence, there exists $k \in \omega(S)$ such that $k \notin F_{\mathsf{Com}}(S)$. This means that $S \in \mathcal{W}_k$ for some k, and there is $i \in I_S$ such that $(\mathsf{Com}i)_k \neq i_S$. Hence, if $i_S = +1$, then $(\mathsf{Com}i)_k = +1$ and $(\mathsf{Com}i)_k \neq +1$, contradiction. If $i_S = -1$, then $(\mathsf{Com}i)_k = -1$, and also $(\mathsf{Com}i)_k \neq -1$, contradiction. ■

Consequently, by virtue of (23) and (45), the kernel of Com is given by:

$$\mathcal{K}(\mathsf{Com}) = \{S \in 2^N \mid \omega(S) \neq \emptyset, \text{ and } S' \subset S \Rightarrow \omega(S') = \emptyset\}. \tag{46}$$

Proposition 3 *Let $j \in N$ and $(N, \mathcal{W}_j)$ be the command game for j. Then for each $k \in N$*

$$Sh_k(N, \mathcal{W}_j) = \psi_{\widetilde{\alpha}^{(Sh)}, \lambda = i_k}(\mathsf{Com}, k \to j) - \psi_{\widetilde{\alpha}^{(Sh)}, \lambda = -i_k}(\mathsf{Com}, k \to j) \tag{47}$$

where $\psi_{\alpha,\lambda}$ is the weighted influence index defined in (25), and for each $i \in I$

$$\widetilde{\alpha}_i^{(Sh)k \to j} = \begin{cases} \frac{1}{n\binom{n-1}{|i|-1}}, & \text{if } i_k = +1 \\ \frac{1}{n\binom{n-1}{n-|i|-1}}, & \text{if } i_k = -1 \end{cases} \tag{48}$$

and $|i| := |\{m \in N \mid i_m = +1\}|$.

Proof: By a coalition S we mean the set of players with the same inclination i_S, while all players outside S have the inclination $-i_S$. Consequently, two coalitions are formed under each inclination vector: a coalition of players with the positive inclination, and a coalition of players with the negative inclination. We have: $k \in S$ iff $i_k = i_S$. Moreover, given $(N, \mathcal{W}_j)$: $v(S) = 1$ iff $S \in \mathcal{W}_j$ iff $(\mathsf{Com} i)_j = i_S$ for each $i \in I_S$ such that $i_m = -i_S$ for each $m \notin S$. We can also define a function $f : 2^N \times \{-1, +1\} \to I$ such that for each $S \subseteq N$ and $i_S \in \{-1, +1\}$, $f(S, i_S) = i$, where $i_m = i_S$ if $m \in S$, and $i_m = -i_S$ if $m \notin S$. From (1), we have

$$\begin{aligned}
Sh_k(N, \mathcal{W}_j) &= \sum_{S \subseteq N : k \in S} \frac{(n-s)!(s-1)!}{n!}(v(S) - v(S \setminus k)) \\
&= \sum_{S \subseteq N : k \in S} \frac{(n-s)!(s-1)!}{n!} v(S) - \sum_{S \subseteq N : k \notin S} \frac{(n-s-1)!s!}{n!} v(S) \\
&= \sum_{S \subseteq N : k \in S} \frac{v(S)}{n\binom{n-1}{s-1}} - \sum_{S \subseteq N : k \notin S} \frac{v(S)}{n\binom{n-1}{n-s-1}} = \frac{1}{2}\left[\sum_{i \in I_k^+} \widetilde{\alpha}_i^{(Sh)k \to j} - \sum_{i \in I_k^-} \widetilde{\alpha}_i^{(Sh)k \to j}\right] \\
&= \frac{\sum_{i \in I_k^+} \widetilde{\alpha}_i^{(Sh)k \to j}}{\sum_{i \in I} \widetilde{\alpha}_i^{(Sh)k \to j}} - \frac{\sum_{i \in I_k^-} \widetilde{\alpha}_i^{(Sh)k \to j}}{\sum_{i \in I} \widetilde{\alpha}_i^{(Sh)k \to j}} \\
&= \psi_{\widetilde{\alpha}^{(Sh)}, \lambda = i_k}(\mathsf{Com}, k \to j) - \psi_{\widetilde{\alpha}^{(Sh)}, \lambda = -i_k}(\mathsf{Com}, k \to j)
\end{aligned}$$

where

$$I_k^+ := \{i \in I \mid (\mathsf{Com} i)_j = i_k\} \tag{49}$$

$$I_k^- := \{i \in I \mid (\mathsf{Com} i)_j = -i_k\} \tag{50}$$

$\psi_{\alpha,\lambda}$ is the weighted influence index defined in (25), and $\widetilde{\alpha}^{(Sh)}$ is given in (48). The last but one equality results from the following facts:

$$\sum_{i : i_k = +1} \widetilde{\alpha}_i^{(Sh)k \to j} = \sum_{i : i_k = +1} \frac{1}{n\binom{n-1}{|i|-1}} = \sum_{|i|=1}^{n} \frac{1}{n\binom{n-1}{|i|-1}} \binom{n-1}{|i|-1} = 1$$

$$\sum_{i : i_k = -1} \widetilde{\alpha}_i^{(Sh)k \to j} = \sum_{i : i_k = -1} \frac{1}{n\binom{n-1}{n-|i|-1}} = \sum_{|i|=0}^{n-1} \frac{1}{n\binom{n-1}{n-|i|-1}} \binom{n-1}{|i|} = 1.$$

■

Proposition 4 *Let $j \in N$ and $(N, \mathcal{W}_j)$ be the command game for j. Then for each $k \in N$*

$$Bz_k(N, \mathcal{W}_j) = \psi_{\widetilde{\alpha},\lambda=i_k}(\mathsf{Com}, k \to j) - \psi_{\widetilde{\alpha},\lambda=-i_k}(\mathsf{Com}, k \to j) \tag{51}$$

where $\psi_{\alpha,\lambda}$ is the weighted influence index defined in (25) with

$$\widetilde{\alpha}_i^{k \to j} = 1, \ \forall i \in I. \tag{52}$$

Proof: The proof is similar to the one of Proposition 3. From (2), we have

$$\begin{aligned} Bz_k(N, \mathcal{W}_j) &= \frac{1}{2^{n-1}} \sum_{S \subseteq N : k \in S} (v(S) - v(S \setminus k)) = \frac{1}{2^{n-1}} \sum_{S \subseteq N : k \in S} v(S) - \frac{1}{2^{n-1}} \sum_{S \subseteq N : k \notin S} v(S) \\ &= \frac{1}{2^{n-1}} \left[\frac{|\{i \in I \mid (\mathsf{Com}i)_j = i_k\}|}{2} - \frac{|\{i \in I \mid (\mathsf{Com}i)_j = -i_k\}|}{2} \right] \\ &= \psi_{\widetilde{\alpha},\lambda=i_k}(\mathsf{Com}, k \to j) - \psi_{\widetilde{\alpha},\lambda=-i_k}(\mathsf{Com}, k \to j) \end{aligned}$$

where $\psi_{\alpha,\lambda}$ is the weighted influence index defined in (25), and $\widetilde{\alpha}$ is given in (52). ■

Proposition 5 *Let $j \in N$ and $(N, \mathcal{W}_j)$ be the command game for j. Then*

$$A(N, \mathcal{W}_j) = \frac{\psi_{\widetilde{\alpha},\lambda\neq 0}(\mathsf{Com}, k \to j)}{2}, \ \forall k \in N \tag{53}$$

and for each $k \in N$

$$Col_k^P(N, \mathcal{W}_j) = \frac{\psi_{\widetilde{\alpha},\lambda=i_k}(\mathsf{Com}, k \to j) - \psi_{\widetilde{\alpha},\lambda=-i_k}(\mathsf{Com}, k \to j)}{\psi_{\widetilde{\alpha},\lambda\neq 0}(\mathsf{Com}, k \to j)} \tag{54}$$

$$KB_k(N, \mathcal{W}_j) = \frac{\psi_{\widetilde{\alpha},\lambda=i_k}(\mathsf{Com}, k \to j)}{\psi_{\widetilde{\alpha},\lambda\neq 0}(\mathsf{Com}, k \to j)} \tag{55}$$

where $\psi_{\alpha,\lambda}$ is the weighted influence index defined in (25) with

$$\widetilde{\alpha}_i^{k \to j} = 1, \ \forall i \in I. \tag{56}$$

Proof: (53) results immediately from (4). Let $j \in N$, and take an arbitrary $k \in N$. We have

$$A(N, \mathcal{W}_j) = \frac{\sum_{S \subseteq N} v(S)}{2^n} = \frac{|\{i \in I \mid (\mathsf{Com}i)_j \neq 0\}|}{2|I|} = \frac{\psi_{\widetilde{\alpha},\lambda\neq 0}(\mathsf{Com}, k \to j)}{2}$$

where $\widetilde{\alpha}$ is given in (56).
We have

$$\sum_{S \subseteq N} v(S) = \frac{|\{i \in I \mid (\mathsf{Com}i)_j \neq 0\}|}{2} = 2^{n-1} \psi_{\widetilde{\alpha},\lambda\neq 0}(\mathsf{Com}, k \to j) > 0. \tag{57}$$

By virtue of (2) and (5), we have for each $j, k \in N$

$$Col_k^P(N, \mathcal{W}_j) = \frac{2^{n-1} Bz_k(N, \mathcal{W}_j)}{\sum_{S \subseteq N} v(S)} = \frac{Bz_k(N, \mathcal{W}_j)}{\psi_{\tilde{\alpha}, \lambda \neq 0}(\mathsf{Com}, k \to j)}$$

which together with (51) gives (54).
Since we have

$$\sum_{S \subseteq N : k \in S} v(S) = \frac{|\{i \in I \mid (\mathsf{Com} i)_j = i_k\}|}{2} = 2^{n-1} \psi_{\tilde{\alpha}, \lambda = i_k}(\mathsf{Com}, k \to j), \tag{58}$$

hence from (7), (57), and (58), we have

$$KB_k(N, \mathcal{W}_j) = \frac{\sum_{S \subseteq N : k \in S} v(S)}{\sum_{S \subseteq N} v(S)} = \frac{\psi_{\tilde{\alpha}, \lambda = i_k}(\mathsf{Com}, k \to j)}{\psi_{\tilde{\alpha}, \lambda \neq 0}(\mathsf{Com}, k \to j)}.$$

■

5.2 An influence function in which a player without a boss follows himself

Next, we mention another command influence function which is compatible with the command games.

Definition 4 *Given the command games $\{(N, \mathcal{W}_k) \mid k \in N\}$, the command influence function $\widetilde{\mathsf{Com}} \in \mathcal{B}$ is defined for each $k \in N$ and $i \in I$ by*

$$(\widetilde{\mathsf{Com}} i)_k := \begin{cases} +1, & \text{if } \{j \in N \mid i_j = +1\} \in \mathcal{W}_k \\ -1, & \text{if } \{j \in N \mid i_j = -1\} \in \mathcal{W}_k \\ i_k, & \text{otherwise} \end{cases}. \tag{59}$$

The command influence function $\widetilde{\mathsf{Com}}$ is similar to the function Com defined in (44) with a difference that for each inclination vector, if none of the two coalitions with all members having the same inclination is winning, now instead of abstaining, the voter in question simply follows his own inclination. As a consequence, we do not have the equality between the sets $\omega(S)$ and $F_{\widetilde{\mathsf{Com}}}(S)$ (as we had before between $\omega(S)$ and $F_{\mathsf{Com}}(S)$), but the inclusion.

Proposition 6 *Let $\{(N, \mathcal{W}_k) \mid k \in N\}$ be the command games, and $\omega(S)$ be the set of all players that are 'commandable' by S, as defined in (11). We have*

$$F_{\widetilde{\mathsf{Com}}}(S) = \omega(S) \cup \{k \in S \mid \forall S' \in \mathcal{W}_k \; [S \cap S' \neq \emptyset]\}, \; \forall S \subseteq N, \tag{60}$$

where $F_{\widetilde{\mathsf{Com}}}(S)$ is the set of followers of S under the command influence function $\widetilde{\mathsf{Com}}$.

Proof: Suppose that $F_{\widetilde{\mathsf{Com}}}(S) \nsubseteq \omega(S) \cup \{k \in S \mid \forall S' \in \mathcal{W}_k\ [S \cap S' \neq \emptyset]\}$ for a certain $S \subseteq N$. Hence, there exists $k \in F_{\widetilde{\mathsf{Com}}}(S)$ such that $k \notin \omega(S)$, and either $k \notin S$ or ($k \in S$, but there is $S' \in \mathcal{W}_k$ such that $S \cap S' = \emptyset$). Consequently, since $k \in F_{\widetilde{\mathsf{Com}}}(S)$ and $k \notin \omega(S)$, we have $(\widetilde{\mathsf{Com}}i)_k = i_S$ for all $i \in I_S$, and $S \notin \mathcal{W}_k$. Take $i \in I_S$ such that $i_S = +1$, and $i_j = -1$ for all $j \notin S$. Hence, $(\widetilde{\mathsf{Com}}i)_k = i_S = +1$. If $k \notin S$, then $(\widetilde{\mathsf{Com}}i)_k = i_S = +1$ and $(\widetilde{\mathsf{Com}}i)_k = -1$, contradiction. Suppose that $k \in S$ and there is $S' \in \mathcal{W}_k$ such that $S \cap S' = \emptyset$. Hence, $i_{S'} = -1$, and therefore $(\widetilde{\mathsf{Com}}i)_k = +1$ and $(\widetilde{\mathsf{Com}}i)_k = -1$, contradiction.
Suppose now that $\omega(S) \cup \{k \in S \mid \forall S' \in \mathcal{W}_k\ [S \cap S' \neq \emptyset]\} \nsubseteq F_{\widetilde{\mathsf{Com}}}(S)$ for a certain $S \subseteq N$. Hence, there exists k such that $k \notin F_{\widetilde{\mathsf{Com}}}(S)$, and either $k \in \omega(S)$ or ($k \in S$ and for all $S' \in \mathcal{W}_k$, $S \cap S' \neq \emptyset$). Since $k \notin F_{\widetilde{\mathsf{Com}}}(S)$, there is $i \in I_S$ such that $(\widetilde{\mathsf{Com}}i)_k = -i_S$. If $k \in \omega(S)$, then $S \in \mathcal{W}_k$, and therefore $(\widetilde{\mathsf{Com}}i)_k = i_S$, contradiction. Suppose $k \in S$ and for all $S' \in \mathcal{W}_k$, $S \cap S' \neq \emptyset$. Hence, in particular, $i_k = i_S$. If there is $S' \in \mathcal{W}_k$ such that $i \in I_{S'}$, then $(\widetilde{\mathsf{Com}}i)_k = i_{S'} = i_S$, contradiction. If $i \notin I_{S'}$ for all $S' \in \mathcal{W}_k$, then $(\widetilde{\mathsf{Com}}i)_k = i_k = i_S$, contradiction. ■

5.3 A non-symmetric influence function

Next, we propose an influence function under which we treat a winning coalition as the set of potential yes-voters only.

Definition 5 *Given the command games $\{(N, \mathcal{W}_k) \mid k \in N\}$, the command influence function $\overline{\mathsf{Com}} \in \mathcal{B}$ is defined by*

$$(\overline{\mathsf{Com}}i)_k := \begin{cases} +1, & \text{if } \{j \in N \mid i_j = +1\} \in \mathcal{W}_k \\ -1, & \text{if } \{j \in N \mid i_j = +1\} \notin \mathcal{W}_k \end{cases}, \quad \forall i \in I\ \forall k \in N. \tag{61}$$

Proposition 7 *Let $\{(N, \mathcal{W}_k) \mid k \in N\}$ be the command games, and $\omega(S)$ be the set of all players that are 'commandable' by S, as defined in (11). We have*

$$F_{\overline{\mathsf{Com}}}(S) = \omega(S),\ \forall S \subseteq N, \tag{62}$$

where $F_{\overline{\mathsf{Com}}}(S)$ is the set of followers of S under the command influence function $\overline{\mathsf{Com}}$.

Proof: $F_{\overline{\mathsf{Com}}}(\emptyset) = \emptyset = \omega(\emptyset)$. Take an arbitrary $S \subseteq N$. Suppose that $F_{\overline{\mathsf{Com}}}(S) \nsubseteq \omega(S)$. Hence, there exists $k \in F_{\overline{\mathsf{Com}}}(S)$ such that $k \notin \omega(S)$, and therefore $(\overline{\mathsf{Com}}i)_k = i_S$ for all $i \in I_S$, and $S \notin \mathcal{W}_k$. Take $i \in I_S$ such that $i_S = +1$. Hence, $(\overline{\mathsf{Com}}i)_k = +1$ and $(\overline{\mathsf{Com}}i)_k = -1$, contradiction.
Suppose now that $\omega(S) \nsubseteq F_{\overline{\mathsf{Com}}}(S)$. Hence, there exists $k \in \omega(S)$ such that $k \notin F_{\overline{\mathsf{Com}}}(S)$. This means that $S \in \mathcal{W}_k$ and there is $i \in I_S$ such that $(\overline{\mathsf{Com}}i)_k = -i_S$. Hence, if $i_S = +1$, then $(\overline{\mathsf{Com}}i)_k = -1$ and $(\overline{\mathsf{Com}}i)_k = +1$, contradiction. If $i_S = -1$, then $(\overline{\mathsf{Com}}i)_k = +1$, and also since $S \in \mathcal{W}_k$, and from (37), $\{j \in N \mid i_j = +1\} \notin \mathcal{W}_k$. Hence $(\overline{\mathsf{Com}}i)_k = -1$ and $(\overline{\mathsf{Com}}i)_k = +1$, contradiction. ■

Consequently, by virtue of (23) and (62), the kernel of $\overline{\mathsf{Com}}$ is given by:

$$\mathcal{K}(\overline{\mathsf{Com}}) = \{S \in 2^N \mid \omega(S) \neq \emptyset, \text{ and } S' \subset S \Rightarrow \omega(S') = \emptyset\}. \tag{63}$$

Proposition 8 *Let $j \in N$ and $(N, \mathcal{W}_j)$ be the command game for j. Then for each $k \in N$*

$$Sh_k(N, \mathcal{W}_j) = \psi_{\widetilde{\alpha}^{(Sh)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j) - \psi_{\widehat{\alpha}^{(Sh)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j) \tag{64}$$

where $\psi_{\alpha,\lambda}$ is the weighted influence index defined in (25), and for each $i \in I$

$$\widetilde{\alpha}_i^{(Sh)k\to j} = \begin{cases} \frac{1}{n\binom{n-1}{|i|-1}}, & \text{if } i_k = +1 \\ 0, & \text{if } i_k = -1 \end{cases} \qquad \widehat{\alpha}_i^{(Sh)k\to j} = \begin{cases} 0, & \text{if } i_k = +1 \\ \frac{1}{n\binom{n-1}{n-|i|-1}}, & \text{if } i_k = -1 \end{cases} \tag{65}$$

and $|i| := |\{m \in N \mid i_m = +1\}|$.

Proof: We introduce a bijection $f : I \to 2^N$ such that for each $i \in I$, $f(i) = \{k \in N \mid i_k = +1\}$. Hence, $k \in S$ iff $i_k = +1$. Given $(N, \mathcal{W}_j)$, we have: $v(S) = 1$ iff $S = f(i) \in \mathcal{W}_j$ iff $(\overline{\mathsf{Com}}i)_j = +1$. Hence, from (1), we have

$$\begin{aligned}
Sh_k(N, \mathcal{W}_j) &= \sum_{S\subseteq N: k\in S} \frac{(n-s)!(s-1)!}{n!}(v(S) - v(S \setminus k)) \\
&= \sum_{S\subseteq N: k\in S} \frac{(n-s)!(s-1)!}{n!}v(S) - \sum_{S\subseteq N: k\notin S} \frac{(n-s-1)!s!}{n!}v(S) \\
&= \sum_{S\subseteq N: k\in S} \frac{v(S)}{n\binom{n-1}{s-1}} - \sum_{S\subseteq N: k\notin S} \frac{v(S)}{n\binom{n-1}{n-s-1}} = \sum_{i\in I_k^+} \widetilde{\alpha}_i^{(Sh)k\to j} - \sum_{i\in I_k^-} \widehat{\alpha}_i^{(Sh)k\to j} \\
&= \frac{\sum_{i\in I_k^+} \widetilde{\alpha}_i^{(Sh)k\to j}}{\sum_{i:i_k=+1} \widetilde{\alpha}_i^{(Sh)k\to j}} - \frac{\sum_{i\in I_k^-} \widehat{\alpha}_i^{(Sh)k\to j}}{\sum_{i:i_k=-1} \widehat{\alpha}_i^{(Sh)k\to j}} \\
&= \psi_{\widetilde{\alpha}^{(Sh)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j) - \psi_{\widehat{\alpha}^{(Sh)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j)
\end{aligned}$$

where

$$I_k^+ := \{i \in I \mid i_k = +1 \ \wedge \ (\overline{\mathsf{Com}}i)_j = +1\} \tag{66}$$

$$I_k^- := \{i \in I \mid i_k = -1 \ \wedge \ (\overline{\mathsf{Com}}i)_j = +1\} \tag{67}$$

$\psi_{\alpha,\lambda}$ is the weighted influence index defined in (25), and $\widetilde{\alpha}^{(Sh)}$ and $\widehat{\alpha}^{(Sh)}$ are given in (65). The last but one equality results from the following facts:

$$\sum_{i:i_k=+1} \widetilde{\alpha}_i^{(Sh)k\to j} = \sum_{i:i_k=+1} \frac{1}{n\binom{n-1}{|i|-1}} = \sum_{|i|=1}^{n} \frac{1}{n\binom{n-1}{|i|-1}} \binom{n-1}{|i|-1} = 1$$

$$\sum_{i:i_k=-1} \widehat{\alpha}_i^{(Sh)k\to j} = \sum_{i:i_k=-1} \frac{1}{n\binom{n-1}{n-|i|-1}} = \sum_{|i|=0}^{n-1} \frac{1}{n\binom{n-1}{n-|i|-1}} \binom{n-1}{|i|} = 1.$$

■

Proposition 9 *Let $j \in N$ and $(N, \mathcal{W}_j)$ be the command game for j. Then for each $k \in N$*

$$Bz_k(N, \mathcal{W}_j) = \psi_{\widetilde{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j) - \psi_{\widehat{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j) \tag{68}$$

where $\psi_{\alpha,\lambda}$ is the weighted influence index defined in (25), and for each $i \in I$

$$\widetilde{\alpha}_i^{(Bz)k\to j} = \begin{cases} 1, & \text{if } i_k = +1 \\ 0, & \text{if } i_k = -1 \end{cases} \qquad \widehat{\alpha}_i^{(Bz)k\to j} = \begin{cases} 0, & \text{if } i_k = +1 \\ 1, & \text{if } i_k = -1. \end{cases} \tag{69}$$

Proof: The proof is similar to the one of Proposition 8. From (2), we have

$$\begin{aligned} Bz_k(N, \mathcal{W}_j) &= \frac{1}{2^{n-1}} \sum_{S\subseteq N: k\in S} (v(S) - v(S \setminus k)) = \frac{1}{2^{n-1}} \sum_{S\subseteq N: k\in S} v(S) - \frac{1}{2^{n-1}} \sum_{S\subseteq N: k\notin S} v(S) \\ &= \frac{|\{i \in I \mid i_k = +1 \ \wedge \ (\overline{\mathsf{Com}}i)_j = +1\}|}{2^{n-1}} - \frac{|\{i \in I \mid i_k = -1 \ \wedge \ (\overline{\mathsf{Com}}i)_j = +1\}|}{2^{n-1}} \\ &= \frac{|\{i \in I \mid i_k = +1 \ \wedge \ (\overline{\mathsf{Com}}i)_j = +1\}|}{|\{i \in I \mid i_k = +1\}|} - \frac{|\{i \in I \mid i_k = -1 \ \wedge \ (\overline{\mathsf{Com}}i)_j = +1\}|}{|\{i \in I \mid i_k = -1\}|} \\ &= \psi_{\widetilde{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j) - \psi_{\widehat{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j) \end{aligned}$$

where $\psi_{\alpha,\lambda}$ is the weighted influence index defined in (25), and $\widetilde{\alpha}^{(Bz)}$ and $\widehat{\alpha}^{(Bz)}$ are given in (69). ■

Proposition 10 *Let $j \in N$ and $(N, \mathcal{W}_j)$ be the command game for j. Then*

$$A(N, \mathcal{W}_j) = \psi_{\widetilde{\alpha},\lambda=+1}(\overline{\mathsf{Com}}, k \to j), \ \forall k \in N \tag{70}$$

and for each $k \in N$

$$Col_k^P(N, \mathcal{W}_j) = \frac{\psi_{\widetilde{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j) - \psi_{\widehat{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j)}{2\psi_{\widetilde{\alpha},\lambda=+1}(\overline{\mathsf{Com}}, k \to j)} \tag{71}$$

$$Col_k^I(N, \mathcal{W}_j) = \frac{\psi_{\widetilde{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j) - \psi_{\widehat{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j)}{2\psi_{\widetilde{\alpha},\lambda=-1}(\overline{\mathsf{Com}}, k \to j)} \tag{72}$$

$$KB_k(N, \mathcal{W}_j) = \frac{\psi_{\widetilde{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j)}{2\psi_{\widetilde{\alpha},\lambda=+1}(\overline{\mathsf{Com}}, k \to j)} \tag{73}$$

where $\psi_{\alpha,\lambda}$ is the weighted influence index defined in (25) with

$$\widetilde{\alpha}_i^{k\to j} = 1, \ \forall i \in I \tag{74}$$

and $\widetilde{\alpha}^{(Bz)}$, $\widehat{\alpha}^{(Bz)}$ are given in (69).

Proof: (70) results immediately from (4). Let $j \in N$, and take an arbitrary $k \in N$. We have

$$A(N, \mathcal{W}_j) = \frac{\sum_{S \subseteq N} v(S)}{2^n} = \frac{|\{i \in I \mid (\overline{\mathsf{Com}}i)_j = +1\}|}{|I|} = \psi_{\widetilde{\alpha},\lambda=+1}(\overline{\mathsf{Com}}, k \to j)$$

where $\widetilde{\alpha}$ is given in (74).
We have

$$\sum_{S \subseteq N} v(S) = |\{i \in I \mid (\overline{\mathsf{Com}}i)_j = +1\}| = 2^n \psi_{\widetilde{\alpha},\lambda=+1}(\overline{\mathsf{Com}}, k \to j) > 0. \tag{75}$$

By virtue of (2) and (5), we have for each $j, k \in N$

$$Col_k^P(N, \mathcal{W}_j) = \frac{2^{n-1} Bz_k(N, \mathcal{W}_j)}{\sum_{S \subseteq N} v(S)} = \frac{Bz_k(N, \mathcal{W}_j)}{2\psi_{\widetilde{\alpha},\lambda=+1}(\overline{\mathsf{Com}}, k \to j)}$$

which together with (68) gives (71).
Moreover we note that

$$\sum_{S \subseteq N : k \in S} v(S) = |\{i \in I \mid i_k = +1 \ \wedge \ (\overline{\mathsf{Com}}i)_j = +1\}| = 2^{n-1} \psi_{\widetilde{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j) \tag{76}$$

$$\sum_{S \subseteq N : k \notin S} v(S) = |\{i \in I \mid i_k = -1 \ \wedge \ (\overline{\mathsf{Com}}i)_j = +1\}| = 2^{n-1} \psi_{\widehat{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j) \tag{77}$$

$$2^n - \sum_{S \subseteq N} v(S) = |\{i \in I \mid (\overline{\mathsf{Com}}i)_j = -1\}| = 2^n \psi_{\widetilde{\alpha},\lambda=-1}(\overline{\mathsf{Com}}, k \to j) \tag{78}$$

By virtue of (6), (76), (77), and (78), we have

$$\begin{aligned}
Col_k^I(N, \mathcal{W}_j) &= \frac{\sum_{S \subseteq N : k \notin S}(v(S \cup k) - v(S))}{2^n - \sum_{S \subseteq N} v(S)} = \frac{\sum_{S \subseteq N : k \in S} v(S) - \sum_{S \subseteq N : k \notin S} v(S)}{2^n - \sum_{S \subseteq N} v(S)} \\
&= \frac{|\{i \in I \mid i_k = +1 \ \wedge \ (\overline{\mathsf{Com}}i)_j = +1\}| - |\{i \in I \mid i_k = -1 \ \wedge \ (\overline{\mathsf{Com}}i)_j = +1\}|}{|\{i \in I \mid (\overline{\mathsf{Com}}i)_j = -1\}|} \\
&= \frac{\psi_{\widetilde{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j) - \psi_{\widehat{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j)}{2\psi_{\widetilde{\alpha},\lambda=-1}(\overline{\mathsf{Com}}, k \to j)}
\end{aligned}$$

From (7), (75), and (76), we have

$$KB_k(N, \mathcal{W}_j) = \frac{\sum_{S \subseteq N : k \in S} v(S)}{\sum_{S \subseteq N} v(S)} = \frac{\psi_{\widetilde{\alpha}^{(Bz)},\lambda=+1}(\overline{\mathsf{Com}}, k \to j)}{2\psi_{\widetilde{\alpha},\lambda=+1}(\overline{\mathsf{Com}}, k \to j)}.$$

■

6 Command games defined for some influence functions

In this section, we work in an opposite direction than in Section 5: for some influence functions defined in [8] we define the command games such that the influence functions are compatible with these games. In particular, we show that not for all influence functions such command games exist.

Proposition 11 *Let $n \geq t > \lfloor \frac{n}{2} \rfloor$ and $\mathsf{Maj}^{[t]} \in \mathcal{B}$ be the majority function as defined in (28). Let $\{(N, \mathcal{W}_k^{\mathsf{Maj}^{[t]}}) \mid k \in N\}$ be the command games given by*

$$\mathcal{W}_k^{\mathsf{Maj}^{[t]}} = \{S \subseteq N \mid s \geq t\}, \ \forall k \in N. \tag{79}$$

The majority function $\mathsf{Maj}^{[t]}$ is compatible with the games $\{(N, \mathcal{W}_k^{\mathsf{Maj}^{[t]}}) \mid k \in N\}$.

Proof: By virtue of (79), the set $\omega^{\mathsf{Maj}^{[t]}}(S)$ of commandable players, for each $S \subseteq N$, is given by

$$\omega^{\mathsf{Maj}^{[t]}}(S) = \begin{cases} N, & \text{if } \ s \geq t \\ \emptyset, & \text{if } \ s < t \end{cases} \tag{80}$$

which from (29) is equal to $F_{\mathsf{Maj}^{[t]}}(S)$. ∎

Remark 1 *Note that for the command games $\{(N, \mathcal{W}_k^{\mathsf{Maj}^{[t]}}) \mid k \in N\}$ defined by (79), we have for $n > 2$, $n > t > \lfloor \frac{n}{2} \rfloor$, and $k \in N$*

$$Boss_k^{\mathsf{Maj}^{[t]}} = \{S \subseteq N \mid s \geq t \ \wedge \ k \notin S\}$$

$$App_k^{\mathsf{Maj}^{[t]}} = \{S \subseteq N \mid s = t-1 \ \wedge \ k \notin S\}.$$

In particular, for $t = n$, $k \in N$,

$$Boss_k^{\mathsf{Maj}^{[t]}} = \emptyset, \quad App_k^{\mathsf{Maj}^{[t]}} = N \setminus k.$$

Proposition 12 *Let $\mathsf{Gur}^{[\widetilde{k}]} \in \mathcal{B}$ be the guru function as defined in (31), with the guru $\widetilde{k} \in N$. Let $\{(N, \mathcal{W}_k^{\mathsf{Gur}^{[\widetilde{k}]}}) \mid k \in N\}$ be the command games given by*

$$\mathcal{W}_k^{\mathsf{Gur}^{[\widetilde{k}]}} = \{S \subseteq N \mid \widetilde{k} \in S\}, \ \forall k \in N. \tag{81}$$

The guru function $\mathsf{Gur}^{[\widetilde{k}]}$ is compatible with the command games $\{(N, \mathcal{W}_k^{\mathsf{Gur}^{[\widetilde{k}]}}) \mid k \in N\}$.

Proof: By virtue of (81), the set $\omega^{\mathsf{Gur}^{[\widetilde{k}]}}(S)$ of commandable players, for each $S \subseteq N$, is given by

$$\omega^{\mathsf{Gur}^{[\widetilde{k}]}}(S) = \begin{cases} N, & \text{if } \ \widetilde{k} \in S \\ \emptyset, & \text{if } \ \widetilde{k} \notin S \end{cases} \tag{82}$$

which from (32) is equal to $F_{\mathsf{Gur}^{[\widetilde{k}]}}(S)$. ∎

Remark 2 *In the command games* $\{(N, \mathcal{W}_k^{\mathsf{Gur}[\widetilde{k}]}) \mid k \in N\}$ *defined by (81), the guru* $\widetilde{k}$ *is a free agent, and the remaining players are cogs, i.e.,*

$$Boss_{\widetilde{k}}^{\mathsf{Gur}[\widetilde{k}]} = \emptyset, \quad App_{\widetilde{k}}^{\mathsf{Gur}[\widetilde{k}]} = 2^{N \setminus \widetilde{k}}$$

$$Boss_k^{\mathsf{Gur}[\widetilde{k}]} = \{S \subseteq N \mid \widetilde{k} \in S \ \wedge \ k \notin S\}, \quad App_k^{\mathsf{Gur}[\widetilde{k}]} = \emptyset, \quad for\ k \neq \widetilde{k}.$$

Proposition 13 *Let* $\mathsf{Id} \in \mathcal{B}$ *be the identity function as defined in (34). Let* $\{(N, \mathcal{W}_k^{\mathsf{Id}}) \mid k \in N\}$ *be the command games given by*

$$\mathcal{W}_k^{\mathsf{Id}} = \{S \subseteq N \mid \ k \in S\}, \ \forall k \in N. \tag{83}$$

The identity function Id *is compatible with the command games* $\{(N, \mathcal{W}_k^{\mathsf{Id}}) \mid k \in N\}$.

Proof: From (83), we have $\omega^{\mathsf{Id}}(S) = S$ for each $S \subseteq N$. On the other hand, also $F_{\mathsf{Id}}(S) = S$ for $S \subseteq N$. ■

Remark 3 *In the command games* $\{(N, \mathcal{W}_k^{\mathsf{Id}}) \mid k \in N\}$ *defined by (83), all players are free agents, i.e.,*

$$Boss_k^{\mathsf{Id}} = \emptyset, \quad App_k^{\mathsf{Id}} = 2^{N \setminus k}.$$

Proposition 14 *Let* $-\mathsf{Id} \in \mathcal{B}$ *be the reversal function as defined in (35). There is no set of command games such that the reversal function is compatible with these command games.*

Proof: We know that $F_{-\mathsf{Id}}(S) = \emptyset$ for $S \subseteq N$. Hence, in particular, $F_{-\mathsf{Id}}(N) = \emptyset$. Suppose there is a set of command games $\{(N, \mathcal{W}_k^{-\mathsf{Id}}) \mid k \in N\}$ such that $-\mathsf{Id}$ is compatible with these command games. We have $\omega(N) = N$ for arbitrary command games, and therefore, we have in particular, $\omega^{-\mathsf{Id}}(N) = N \not\subseteq F_{-\mathsf{Id}}(N)$, contradiction. ■

7 Relation between the influence model and the command games

7.1 Influence functions and followers

Any influence function B is a mapping from I to I, hence the cardinality of $\mathcal{B}$ is $(2^n)^{(2^n)} = 2^{n2^n}$. The 'follower function' of B, denoted by F_B, is a mapping from 2^N to 2^N. Hence there are $(2^n)^{(2^n)}$ such functions, as many as influence functions. Let us denote by $\mathcal{F}$ the set of mappings from 2^N to 2^N. However, while there is no restriction on B, F_B should satisfy some conditions, like monotonicity. Hence there are functions in $\mathcal{F}$ which cannot correspond to the follower function of some influence function, and consequently, several B's may have the same follower function (put differently, we loose some information by considering only F_B). Formally, this means that the mapping $\Phi : \mathcal{B} \to \mathcal{F}$, defined by $B \mapsto \Phi(B) := F_B$, is neither a surjection nor an injection. This raises the following natural questions:

– Given a function in $\mathcal{F}$, which are the sufficient and necessary conditions so that there exists $B \in \mathcal{B}$ such that $F \equiv F_B$? Let us call F a *follower function* if these conditions are satisfied.
– If F is indeed a follower function, can we find a B such that $F_B \equiv F$?

The following results answer the questions.

Proposition 15 *A function $F \in \mathcal{F}$ is a follower function for some $B \in \mathcal{B}$ (i.e., $F_B \equiv F$, or $\Phi(B) = F$) if and only if it satisfies the following three conditions:*

(i) $F(\emptyset) = \emptyset$;
(ii) F *is monotone, i.e.,* $S \subseteq T$ *implies* $F(S) \subseteq F(T)$;
(iii) *if* $S \cap T = \emptyset$, *then* $F(S) \cap F(T) = \emptyset$.

Moreover, the smallest and greatest influence functions belonging to $\Phi^{-1}(F)$ are respectively the influence functions $\underline{B}_F$ and $\overline{B}_F$, for all $i \in I$ and all $k \in N$ defined by:

$$(\underline{B}_F i)_k := \begin{cases} +1, & \text{if } k \in F(S^+(i)) \\ -1, & \text{else} \end{cases}$$

$$(\overline{B}_F i)_k := \begin{cases} -1, & \text{if } k \in F(S^-(i)) \\ +1, & \text{else} \end{cases}$$

where we put for convenience $S^{\pm}(i) := \{j \in N \mid i_j = \pm 1\}$. We call these influence functions the upper and lower inverses of F.

Proof: We already know from [8, Prop. 2] that any follower function fulfills the above three conditions.

Take $F \in \mathcal{F}$ satisfying the above conditions. Let us check if indeed $\Phi(\underline{B}_F) =: F_{\underline{B}_F} = F$. We have to prove that $F_{\underline{B}_F}(S) = F(S)$ for all $S \subseteq N$. It is true for $S = \emptyset$, by definition of follower functions, and the condition $F(\emptyset) = \emptyset$.

We consider some subset $S \neq \emptyset$. Let us first study the case where $F(S) = \emptyset$. This implies that $\underline{B}_F i = (-1, \dots, -1)$, for $i = (1_S, -1_{N \setminus S})$, which in turn implies that $F_{\underline{B}_F}(S) = \emptyset$.

Suppose now $F(S) \neq \emptyset$, and $k \in F(S)$. Let us show that $k \in F_{\underline{B}_F}(S)$. For $i = (1_S, -1_{N \setminus S}) \in I_S$, we have $(\underline{B}_F i)_k = 1 = i_S$. We have to show that this remains true for any $i \in I_S$. We have $I_S = \{(1_{S'}, -1_{N \setminus S'}) \mid S' \supseteq S\} \cup \{(-1_{S'}, 1_{N \setminus S'}) \mid S' \supseteq S\}$. If $i = (1_{S'}, -1_{N \setminus S'})$ for $S' \supseteq S$, we have $(\underline{B}_F i)_k = i_S = 1$ if $k \in F(S')$, which is true since $k \in F(S)$ and F is monotone. If $i = (-1_{S'}, 1_{N \setminus S'})$ for $S' \supseteq S$, we have $(\underline{B}_F i)_k = i_S = -1$ if $k \notin F(N \setminus S')$. Since $S \cap (N \setminus S') = \emptyset$, by the third condition we have $F(S) \cap F(N \setminus S') = \emptyset$, hence $k \notin F(N \setminus S')$. In conclusion, $k \in F_{\underline{B}_F}(S)$.

Conversely, if $F_{\underline{B}_F}(S) = \emptyset$, then necessarily $F(S) = \emptyset$ too, since we have proved above that any element in $F(S)$ is also in $F_{\underline{B}_F}(S)$. Suppose now that $F_{\underline{B}_F}(S) \neq \emptyset$, and take $k \in F_{\underline{B}_F}(S)$. Then for any $i \in I_S$, $(\underline{B}_F i)_k = i_S$. In particular, $i := (1_S, -1_{N \setminus S}) \in I_S$, so that $(\underline{B}_F i)_k = 1$, which implies that $k \in F(S)$.

Finally, $\underline{B}_F$ is the smallest B such that $\Phi(B) = F$ because any B in $\Phi^{-1}(F)$ must satisfy for any $k \in F(S) \neq \emptyset$, $Bi_k = 1$ for $i = (1_S, -1_{N \setminus S})$. Hence $B \geq \underline{B}_F$.

The proof for the upper inverse is much the same. ■

Example 1 Consider $F \equiv \emptyset$, which is a follower function. We know already from [8, Prop. 7] that an inverse of F is the reversal function $-\mathsf{Id}$. Clearly, the lower inverse is the constant function $B \equiv (-1, \dots, -1)$, while the upper inverse is $B \equiv (1, \dots, 1)$.

Example 2 Consider $F = Id$, which is a follower function. We know already from [8, Prop. 6] that an inverse of F is the identity function Id. Clearly, the lower and upper inverses collapse to Id. Hence, $\Phi^{-1}(Id) = \{\mathsf{Id}\}$.

Remark 4 *Denoting inclination vectors by their corresponding subset of players with positive inclination, the above results can be written in a simpler way. It is easy to see that*

$$\underline{B}_F(S) = F(S), \quad \overline{B}_F(S) = \overline{F(\overline{S})}.$$

Also, the definition of the follower function can be rewritten as

$$F(S) = \bigcap_{S' \supseteq S} B(S') \cap \bigcap_{S' \subseteq N \setminus S} \overline{B(S')}.$$

The following example uses this notation.

Example 3 Let $n = 3$ and the following function F be defined as follows:

S	$\emptyset$	1	2	3	12	13	23	123
$F(S)$	$\emptyset$	$\emptyset$	2	$\emptyset$	2	3	12	123

Then the upper and lower inverses are

i	$\emptyset$	1	2	3	12	13	23	123
$\overline{B}_F i$	$\emptyset$	3	12	13	123	13	123	123
$\underline{B}_F i$	$\emptyset$	$\emptyset$	2	$\emptyset$	2	3	12	123

7.2 Command games and commandable players

Similar questions can be raised concerning command games and the set of commandable players, as well as the exact relationship between command games and influence functions. We start by studying the relation between command games and commandable players.

We begin by some simple (but useful) observations. A set of command games $(\mathcal{W}_k, N)$ for $k \in N$ can be viewed more compactly as a mapping $\Omega : N \times 2^N \rightarrow \{0, 1\}$, with $(k, S) \mapsto \Omega(k, S) = 1$ if $S \in \mathcal{W}_k$, and 0 else. Let us call $\mathcal{G}$ the set of such functions. Hence the cardinality of $\mathcal{G}$ is 2^{n2^n}, which is exactly the cardinality of $\mathcal{B}$. What are the restrictions imposed on the command games? If we examine the structure induced by boss and approval sets, we find that $\mathcal{W}_k$ is a union of principal filters, and hence is an upset: $\mathcal{W}_k = \uparrow S_1 \cup \ldots \cup \uparrow S_l$, produced either by minimal boss sets (those S_j not containing k) or by approval sets (those S_j containing k)[1,2]. Now, another constraint may be to impose that $S_1 \cap \cdots \cap S_l \neq \emptyset$ for each $\mathcal{W}_k$. This implies in particular that there is no two disjoint boss sets. Lastly, the empty set cannot be a boss set (this does not make sense), hence $\mathcal{W}_k \neq \uparrow \emptyset = 2^N$. This leads to the following definition.

[1] A family of subsets is an upset if any superset of an element of the family belongs also to te family. For any $S \subseteq N$, we define $\uparrow S := \{T \subseteq N \mid T \supseteq S\}$, the *principal filter* of S. Evidently, for N being finite, any upset is a union of principal filters.

[2] It is not written explicitly if $\mathcal{W}_k$ could be empty, but since $\omega(N) = N$ is assumed, necessarily $\mathcal{W}_k \ni N$ for all $k \in N$.

Definition 6 *A normal command game Ω is a set of simple games $(\mathcal{W}_k, N)_{k\in N}$ satisfying the two conditions:*

(i) For each $k \in N$, there exists a nonempty family of nonempty subsets $S_1^k, \dots, S_{l_k}^k$ (called the minimal sets generating *$\mathcal{W}_k$) such that $\mathcal{W}_k = \uparrow S_1^k \cup \ldots \cup \uparrow S_{l_k}^k$.*

(ii) For each $k \in N$, $S_1^k \cap \cdots \cap S_{l_k}^k \neq \emptyset$.

Next we consider $\omega(S)$, the set of players commandable by S. Clearly $w : 2^N \to 2^N$, so the set of all such mappings, which is $\mathcal{F}$, has the same cardinality as the set of functions Ω. There exists an obvious bijection between $\mathcal{G}$ and $\mathcal{F}$, let us call it Ψ. We have

$$\Psi(\Omega) = \omega, \quad \text{with } \omega(S) = \{k \in N \mid S \in \mathcal{W}_k\}, \quad \forall S \subseteq N$$
$$\Psi^{-1}(\omega) = \Omega, \quad \text{with } S \in \mathcal{W}_k \text{ iff } k \in \omega(S).$$

Proposition 16 *Let $\omega \in \mathcal{F}$. Then ω corresponds to some normal command game if and only if the following conditions are satisfied:*

(i) $\omega(\emptyset) = \emptyset$, $\omega(N) = N$;

(ii) ω is monotone w.r.t. set inclusion;

(iii) If $S \cap S' = \emptyset$, then $\omega(S) \cap \omega(S') = \emptyset$.

Proof: Suppose ω corresponds to some normal command game. Then $\omega(\emptyset) = \emptyset$ follows from the fact that $\emptyset \notin \mathcal{W}_k$, $\forall k \in N$. On the other hand, $\omega(N) = N$ since $N \in \mathcal{W}_k$, $\forall k \in N$. Next, take $S \subseteq S' \subseteq N$. If $k \in \omega(S)$, then $k \in \omega(S')$ too due to the definition of $\mathcal{W}_k$, which proves that $\omega(S) \subseteq \omega(S')$. Lastly, if $k \in \omega(S) \cap \omega(S')$, then both S, S' belong to $\mathcal{W}_k$, and so they must have a nonempty intersection.

Conversely, assume that ω fulfills the three conditions, and consider the corresponding Ω. Since $\omega(N) = N$, each $\mathcal{W}_k$ contains N, and thus is nonempty. Since $\omega(\emptyset) = \emptyset$, no $\mathcal{W}_k$ contains the emptyset. Take any $\mathcal{W}_k$, and consider $S \in \mathcal{W}_k$. Then any $S' \supseteq S$ belongs also to $\mathcal{W}_k$, since $S \subseteq S'$ implies $\omega(S) \subseteq \omega(S')$. This proves that $\mathcal{W}_k$ is an upset, hence it is a union of principal filters $\uparrow S_1^k, \dots, \uparrow S_{l_k}^k$. It remains to prove that there is no pair of disjoint sets in this family. Assuming $\mathcal{W}_k$ contains at least two subsets (otherwise the condition is void), take $S, S' \in \mathcal{W}_k$ such that $S \cap S' = \emptyset$. Then by (iii), $\omega(S) \cap \omega(S') = \emptyset$, which contradicts that fact that $S, S' \in \mathcal{W}_k$. ■

If $\omega \in \mathcal{F}$ satisfies the conditions of Prop. 16, the notion of kernel is meaningful. We denote it by $\mathcal{K}(\omega)$.

Proposition 17 *Let $\omega \in \mathcal{F}$ satisfy the conditions of Prop. 16. Then the generating family $S_1^k, \dots, S_{l_k}^k$ of $\mathcal{W}_k$ is given by:*

(i) If there exists $S \in \mathcal{K}(\omega)$ such that $k \in S$:

$$\{S_1^k, \dots, S_{l_k}^k\} = \{S \in \mathcal{K}(\omega) \mid \omega(S) \ni k\};$$

(ii) Otherwise:

$$\{S_1^k, \dots, S_{l_k}^k\} = \{S \in 2^N \mid \omega(S) \ni k \text{ and } S' \subset S \Rightarrow \omega(S') \not\ni k\}.$$

Proof: Clear. ■

7.3 Influence functions and command games

We turn to the relation between influence functions and command games.

Definition 7 *Let B be an influence function and Ω be a command game. Then B and Ω are equivalent if $F_B \equiv \omega$. They are compatible if $\omega(S) \subseteq F_B(S)$, for all $S \subseteq N$.*

Due to the previous results, the case of equivalence is elucidated.

Theorem 1 *(i) Let B be an influence function. Then B is equivalent to a (unique) normal command game Ω if and only if $F_B(N) = N$. The generating families $S_1^k, \dots, S_{l_k}^k$, $k \in N$, of Ω are given by Prop. 17, which gives the minimal boss sets and approval sets:*

$$\mathsf{Boss}_k^* = \{S_j^k \mid S_j^k \not\ni k, j = 1, \dots, l_k\}, \quad \mathsf{App}_k^* = \{S_j^k \mid S_j^k \ni k, j = 1, \dots, l_k\}.$$

(ii) Let Ω be a normal command game. Then any influence function in $\Phi^{-1}(\omega)$ is equivalent to Ω, in particular the upper and lower inverse $\overline{B}_\omega$ and $\underline{B}_\omega$. Moreover, the kernel of any influence function B in $\Phi^{-1}(\omega)$ is given by

$$\mathcal{K}(B) = \min\Big(\bigcup_{k \in N} \{S_1^k, \dots, S_{l_k}^k\}\Big)$$

where $min(\dots)$ means that only minimal sets are selected from the collection.

8 Example - The Confucian model

In [13] the Confucian model of society is mentioned. We have four players in the society, i.e., $N = \{1, 2, 3, 4\}$ with the king (1), the man (2), the wife (3), and the child (4). The rules are as follows:

(i) The man follows the king;
(ii) The wife and the child follow the man;
(iii) The king should respect his people.

Let us define the command games for this example. By virtue of the rules (i) and (ii), we have immediately:

$$\mathcal{W}_2 = \{1, 12, 13, 14, 123, 124, 134, 1234\}$$
$$\mathcal{W}_3 = \{2, 12, 23, 24, 123, 124, 234, 1234\}$$
$$\mathcal{W}_4 = \{2, 12, 23, 24, 123, 124, 234, 1234\}.$$

Hence, we have

$$Boss_2 = \{1, 13, 14, 134\}, \quad Boss_3 = \{2, 12, 24, 124\}, \quad Boss_4 = \{2, 12, 23, 123\}$$

$$App_2 = App_3 = App_4 = \emptyset,$$

which means that players 2, 3, and 4 are the cogs.
How can we translate the rule (iii) into the set $\mathcal{W}_1$ of winning coalitions in the command game for player 1? We propose several interpretations of this rule, and consequently, several command games for player 1.

8.1 The command game with $\mathcal{W}_1 = \{1234\}$

If $\mathcal{W}_1 = \{1234\}$, then we get:

$$Boss_1 = \emptyset, \quad App_1 = \{234\},$$

i.e., the king needs the approval of all his people.

$$\omega(1) = \omega(13) = \omega(14) = \omega(134) = \{2\}, \quad \omega(2) = \omega(23) = \omega(24) = \omega(234) = \{3,4\}$$

$$\omega(3) = \omega(4) = \omega(34) = \emptyset, \quad \omega(N) = N, \quad \omega(12) = \omega(123) = \omega(124) = \{2,3,4\}.$$

The Shapley-Shubik index matrix is then:

$$P = Sh = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and from (13) one has

$$\begin{cases} \pi_1 = \frac{1}{4}\pi_1 + \pi_2 \\ \pi_2 = \frac{1}{4}\pi_1 + \pi_3 + \pi_4 \\ \pi_3 = \frac{1}{4}\pi_1 \\ \pi_4 = \frac{1}{4}\pi_1 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

which gives the authority distribution:

$$\pi = \frac{1}{9}(4,3,1,1).$$

Let us apply now the command influence functions to the model. Table 1 presents the inclination and decision vectors under the three command influence functions.

Table 1. The inclination and decision vectors for $\mathcal{W}_1 = \{1234\}$

$i \in I$	$\mathsf{Com}i$	$\widetilde{\mathsf{Com}}i$	$\overline{\mathsf{Com}}i$
$(1,1,1,1)$	$(1,1,1,1)$	$(1,1,1,1)$	$(1,1,1,1)$
$(1,1,1,-1)$	$(0,1,1,1)$	$(1,1,1,1)$	$(-1,1,1,1)$
$(1,1,-1,1)$	$(0,1,1,1)$	$(1,1,1,1)$	$(-1,1,1,1)$
$(1,-1,1,1)$	$(0,1,-1,-1)$	$(1,1,-1,-1)$	$(-1,1,-1,-1)$
$(-1,1,1,1)$	$(0,-1,1,1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$
$(1,-1,1,-1)$	$(0,1,-1,-1)$	$(1,1,-1,-1)$	$(-1,1,-1,-1)$
$(1,1,-1,-1)$	$(0,1,1,1)$	$(1,1,1,1)$	$(-1,1,1,1)$
$(-1,1,1,-1)$	$(0,-1,1,1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$
$(1,-1,-1,1)$	$(0,1,-1,-1)$	$(1,1,-1,-1)$	$(-1,1,-1,-1)$
$(-1,-1,1,1)$	$(0,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$
$(-1,1,-1,1)$	$(0,-1,1,1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$
$(1,-1,-1,-1)$	$(0,1,-1,-1)$	$(1,1,-1,-1)$	$(-1,1,-1,-1)$
$(-1,1,-1,-1)$	$(0,-1,1,1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$
$(-1,-1,1,-1)$	$(0,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$
$(-1,-1,-1,1)$	$(0,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$
$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$

Note that under $B \in \{\mathsf{Com}, \widetilde{\mathsf{Com}}, \overline{\mathsf{Com}}\}$ each player (except the king) always follows the inclination of his boss sets. Under the influence function Com, the king will say 'yes' / 'no' only if all players (including himself) have the positive / negative inclination. In the remaining cases, the king abstains. Under the influence function $\widetilde{\mathsf{Com}}$, the king always follows his own inclination, since 1234 is the unique winning coalition in the command game for player 1. Under the influence function $\overline{\mathsf{Com}}$, if all people of the king are against, his decision is also 'no', but if all his people are in favor (i.e., the king has the approval of his people), his decision is 'yes' only if his inclination is also positive. In case the people of the king are not unanimous, the king has no approval, and consequently he chooses 'no', even if his inclination is positive.

We have, in particular:

$$F_B(1) = \{2\}, \quad F_B(2) = \{3,4\}, \quad F_B(12) = \{2,3,4\}, \quad for\ B \in \{\mathsf{Com}, \overline{\mathsf{Com}}\}$$

$$F_{\widetilde{\mathsf{Com}}}(1) = \{1,2\}, \quad F_{\widetilde{\mathsf{Com}}}(2) = \{3,4\}, \quad F_{\widetilde{\mathsf{Com}}}(12) = N.$$

By virtue of (22), $d_\alpha(B, S \to j) = 1$ for each $j \in F_B(S) \setminus S$. Hence, we have:

$$d_\alpha(B, 1 \to 2) = d_\alpha(B, 2 \to 3) = d_\alpha(B, 2 \to 4) = 1, \quad for\ B \in \{\mathsf{Com}, \widetilde{\mathsf{Com}}, \overline{\mathsf{Com}}\}.$$

The kernel is equal to:

$$\mathcal{K}(B) = \{\{1\}, \{2\}\}, \quad for\ B \in \{\mathsf{Com}, \widetilde{\mathsf{Com}}, \overline{\mathsf{Com}}\}.$$

We have already calculated the Shapley-Shubik index matrix for this example. The Banzhaf index matrices are equal to:

$$Bz = \begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \widetilde{Bz} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The Coleman matrix A is equal to:

$$A = [A(N, \mathcal{W}_j)]_{j \in N} = \left[\frac{1}{16}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right].$$

Moreover, we have

$$Col^P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad Col^I = \begin{bmatrix} \frac{1}{15} & \frac{1}{15} & \frac{1}{15} & \frac{1}{15} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad KB = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

8.2 The command game with $\mathcal{W}_1 = \{123, 124, 134, 1234\}$

If $\mathcal{W}_1 = \{123, 124, 134, 1234\}$, then we get:

$$Boss_1 = \emptyset, \quad App_1 = \{23, 24, 34, 234\},$$

i.e., the king needs the approval of the majority of his people.

$$\omega(1) = \omega(13) = \omega(14) = \{2\}, \quad \omega(134) = \{1,2\}, \quad \omega(2) = \omega(23) = \omega(24) = \omega(234) = \{3,4\}$$

$$\omega(3) = \omega(4) = \omega(34) = \emptyset, \quad \omega(12) = \{2,3,4\}, \quad \omega(123) = \omega(124) = \omega(N) = N.$$

The Shapley-Shubik index matrix is then:

$$P = Sh = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and from (13) one has

$$\begin{cases} \pi_1 = \frac{1}{2}\pi_1 + \pi_2 \\ \pi_2 = \frac{1}{6}\pi_1 + \pi_3 + \pi_4 \\ \pi_3 = \frac{1}{6}\pi_1 \\ \pi_4 = \frac{1}{6}\pi_1 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

which gives the authority distribution:

$$\pi = \frac{1}{11}(6, 3, 1, 1).$$

As before, we apply now the command influence functions to the model. Table 2 presents the inclination and decision vectors under the three command influence functions.

Table 2. The inclination and decision vectors for $\mathcal{W}_1 = \{123, 124, 134, 1234\}$

$i \in I$	$\mathsf{Com}i$	$\widetilde{\mathsf{Com}}i$	$\overline{\mathsf{Com}}i$
$(1,1,1,1)$	$(1,1,1,1)$	$(1,1,1,1)$	$(1,1,1,1)$
$(1,1,1,-1)$	$(1,1,1,1)$	$(1,1,1,1)$	$(1,1,1,1)$
$(1,1,-1,1)$	$(1,1,1,1)$	$(1,1,1,1)$	$(1,1,1,1)$
$(1,-1,1,1)$	$(1,1,-1,-1)$	$(1,1,-1,-1)$	$(1,1,-1,-1)$
$(-1,1,1,1)$	$(0,-1,1,1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$
$(1,-1,1,-1)$	$(0,1,-1,-1)$	$(1,1,-1,-1)$	$(-1,1,-1,-1)$
$(1,1,-1,-1)$	$(0,1,1,1)$	$(1,1,1,1)$	$(-1,1,1,1)$
$(-1,1,1,-1)$	$(0,-1,1,1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$
$(1,-1,-1,1)$	$(0,1,-1,-1)$	$(1,1,-1,-1)$	$(-1,1,-1,-1)$
$(-1,-1,1,1)$	$(0,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$
$(-1,1,-1,1)$	$(0,-1,1,1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$
$(1,-1,-1,-1)$	$(0,1,-1,-1)$	$(1,1,-1,-1)$	$(-1,1,-1,-1)$
$(-1,1,-1,-1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$
$(-1,-1,1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$
$(-1,-1,-1,1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$
$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$

Note that for $\mathcal{W}_1 = \{123, 124, 134, 1234\}$ the decision vectors under $\widetilde{\mathsf{Com}}$ are the same as the decision vectors under $\widetilde{\mathsf{Com}}$ for $\mathcal{W}_1 = \{1234\}$.

The results are similar as before, that is:

$$F_B(1) = \{2\}, \quad F_B(2) = \{3,4\}, \quad F_B(12) = \{2,3,4\}, \quad for\ B \in \{\mathsf{Com}, \overline{\mathsf{Com}}\}$$

$$F_{\widetilde{\mathsf{Com}}}(1) = \{1,2\}, \quad F_{\widetilde{\mathsf{Com}}}(2) = \{3,4\}, \quad F_{\widetilde{\mathsf{Com}}}(12) = N.$$

In particular:

$$d_\alpha(B, 1 \to 2) = d_\alpha(B, 2 \to 3) = d_\alpha(B, 2 \to 4) = 1, \quad for\ B \in \{\mathsf{Com}, \widetilde{\mathsf{Com}}, \overline{\mathsf{Com}}.\}$$

The kernel is equal to:

$$\mathcal{K}(B) = \{\{1\},\{2\}\}, \quad for\ B \in \{\mathsf{Com}, \widetilde{\mathsf{Com}}, \overline{\mathsf{Com}}\}.$$

The Banzhaf index matrices are equal to:

$$Bz = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \widetilde{Bz} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The Coleman matrix A is equal to:

$$A = [A(N, \mathcal{W}_j)]_{j \in N} = \left[\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right].$$

Moreover, we have

$$Col^P = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad Col^I = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad KB = \begin{bmatrix} 1 & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

8.3 The command game with $\mathcal{W}_1 = \{12, 13, 14, 123, 124, 134, 1234\}$

If $\mathcal{W}_1 = \{12, 13, 14, 123, 124, 134, 1234\}$, then we get:

$$Boss_1 = \emptyset, \quad App_1 = \{2, 3, 4, 23, 24, 34, 234\},$$

i.e., the king needs the approval of at least one of his people.

$$\omega(1) = \{2\}, \quad \omega(13) = \omega(14) = \omega(134) = \{1,2\}, \quad \omega(2) = \omega(23) = \omega(24) = \omega(234) = \{3,4\}$$

$$\omega(3) = \omega(4) = \omega(34) = \emptyset, \quad \omega(12) = \omega(123) = \omega(124) = \omega(N) = N.$$

The Shapley-Shubik index matrix is then:

$$P = Sh = \begin{bmatrix} \frac{3}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and from (13) one has

$$\begin{cases} \pi_1 = \frac{3}{4}\pi_1 + \pi_2 \\ \pi_2 = \frac{1}{12}\pi_1 + \pi_3 + \pi_4 \\ \pi_3 = \frac{1}{12}\pi_1 \\ \pi_4 = \frac{1}{12}\pi_1 \\ \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1 \end{cases}$$

which gives the authority distribution:

$$\pi = \frac{1}{17}(12, 3, 1, 1).$$

We apply the command influence functions to the model. Table 3 presents the inclination and decision vectors under the command influence functions.

Table 3. The inclination and decision vectors for $\mathcal{W}_1 = \{12, 13, 14, 123, 124, 134, 1234\}$

$i \in I$	$\mathsf{Com}i$	$\widetilde{\mathsf{Com}}i$	$\overline{\mathsf{Com}}i$
$(1,1,1,1)$	$(1,1,1,1)$	$(1,1,1,1)$	$(1,1,1,1)$
$(1,1,1,-1)$	$(1,1,1,1)$	$(1,1,1,1)$	$(1,1,1,1)$
$(1,1,-1,1)$	$(1,1,1,1)$	$(1,1,1,1)$	$(1,1,1,1)$
$(1,-1,1,1)$	$(1,1,-1,-1)$	$(1,1,-1,-1)$	$(1,1,-1,-1)$
$(-1,1,1,1)$	$(0,-1,1,1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$
$(1,-1,1,-1)$	$(1,1,-1,-1)$	$(1,1,-1,-1)$	$(1,1,-1,-1)$
$(1,1,-1,-1)$	$(1,1,1,1)$	$(1,1,1,1)$	$(1,1,1,1)$
$(-1,1,1,-1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$
$(1,-1,-1,1)$	$(1,1,-1,-1)$	$(1,1,-1,-1)$	$(1,1,-1,-1)$
$(-1,-1,1,1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$
$(-1,1,-1,1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$
$(1,-1,-1,-1)$	$(0,1,-1,-1)$	$(1,1,-1,-1)$	$(-1,1,-1,-1)$
$(-1,1,-1,-1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$	$(-1,-1,1,1)$
$(-1,-1,1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$
$(-1,-1,-1,1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$
$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-1)$

As before, under all three influence functions, each player (except the king) always follows the inclination of his boss sets. Moreover, under Com, the king almost always follows his inclination, except two cases of abstention when the inclination of all his people is different from his own inclination. Under the function $\widetilde{\mathsf{Com}}$, the king always follows his own inclination as in the previous two command games for player 1. Under the function $\overline{\mathsf{Com}}$, the king almost always follows his inclination, except one case when his inclination is positive while the inclination of all his people is negative.

We have also:

$$F_B(1) = \{2\}, \quad F_B(2) = \{3,4\}, \quad F_B(12) = N, \quad for\ B \in \{\mathsf{Com}, \overline{\mathsf{Com}}\}$$
$$F_{\widetilde{\mathsf{Com}}}(1) = \{1,2\}, \quad F_{\widetilde{\mathsf{Com}}}(2) = \{3,4\}, \quad F_{\widetilde{\mathsf{Com}}}(12) = N.$$

$$d_\alpha(B, 1 \to 2) = d_\alpha(B, 2 \to 3) = d_\alpha(B, 2 \to 4) = 1, \quad for\ B \in \{\mathsf{Com}, \widetilde{\mathsf{Com}}, \overline{\mathsf{Com}}.\}$$

$$\mathcal{K}(B) = \{\{1\},\{2\}\}, \quad for\ B \in \{\mathsf{Com}, \widetilde{\mathsf{Com}}, \overline{\mathsf{Com}}\}.$$

The Banzhaf index matrices are equal to:

$$Bz = \begin{bmatrix} \frac{7}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \widetilde{Bz} = \begin{bmatrix} \frac{7}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The Coleman matrix A is equal to:

$$A = [A(N, \mathcal{W}_j)]_{j \in N} = \left[\frac{7}{16}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right].$$

Moreover, we have

$$Col^P = \begin{bmatrix} 1 & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad Col^I = \begin{bmatrix} \frac{7}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad KB = \begin{bmatrix} 1 & \frac{4}{7} & \frac{4}{7} & \frac{4}{7} \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

9 Conclusion

This paper presents a comparison between two models that deal with modeling players' interactions: the command games and the influence model. The link between the models is expressed by defining the influence functions that are compatible with the command games, in the sense that each commandable player for a coalition in the command game is a follower of the coalition under the command influence function. Consequently, given a set of the command games, we define three influence functions that are compatible with the command games. For some influence functions we define the command games such that the influence functions are compatible with these games. Nevertheless, not for all influence functions such command games exist. In the paper, we also show links between some power indices, which can be used in the command games, and the generalized weighted influence indices. The concluding remark can be that the influence model is more general than the framework of the command games, and the concepts of the influence function and the influence index can capture the command structure.

A research agenda concerning our future work on the influence model contains several issues. In particular, we plan to introduce the authority distribution based on the influence indices. Moreover, we intend to introduce dynamic aspects into the model. We want to study the behavior of the series $Bi, B^2i, ..., B^ni, ...$, to find convergence conditions, to investigate the corresponding influence indices, and relations between the repeated influence model and the command games. We plan to analyze a generalized model of influence, in which each player has a continuum of options to choose (that is, a model in which the inclination or opinion of each player lies in an interval, say $[0, 1]$, where each $i_k \in [0, 1]$ can be interpreted as player k's degree of inclination to say 'yes'). An important issue for future research concerns an axiomatic characterization of the influence indices. Furthermore, it would be interesting to test the new concepts and to run lab experiments concerning the influence between players.

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